

HOM-LIE ADMISSIBLE HOM-COALGEBRAS AND HOM-HOPF ALGEBRAS

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ABSTRACT. The aim of this paper is to generalize the concept of Lie-admissible coalgebra introduced in [2] to Hom-coalgebras and to introduce Hom-Hopf algebras with some properties. These structures are based on the Hom-algebra structures introduced in [12].

INTRODUCTION

In [4, 7, 8], the class of quasi-Lie algebras and subclasses of quasi-hom-Lie algebras and Hom-Lie algebras has been introduced. These classes of algebras are tailored in a way suitable for simultaneous treatment of the Lie algebras, Lie superalgebras, the color Lie algebras and the deformations arising in connection with twisted, discretized or deformed derivatives and corresponding generalizations, discretizations and deformations of vector fields and differential calculus. It has been shown in [4, 7, 8, 9] that the class of quasi-Hom-Lie algebras contains as a subclass on the one hand the color Lie algebras and in particular Lie superalgebras and Lie algebras, and on the another hand various known and new single and multi-parameter families of algebras obtained using twisted derivations and constituting deformations and quasi-deformations of universal enveloping algebras of Lie and color Lie algebras and of algebras of vector-fields. The main feature of quasi-Lie algebras, quasi-Hom-Lie algebras and Hom-Lie algebras is that the skew-symmetry and the Jacobi identity are twisted by several deforming twisting maps and also in quasi-Lie and quasi-Hom-Lie algebras the Jacobi identity in general contains 6 twisted triple bracket terms.

In the paper [12], we provided a different way for constructing Hom-Lie algebras by extending the fundamental construction of Lie algebras from associative algebras via commutator bracket multiplication. To this end we defined the notion of Hom-associative algebras generalizing associative algebras to a situation where associativity law is twisted, and showed that the commutator product defined using the multiplication in a Hom-associative algebra leads naturally to Hom-Lie algebras. We introduced also Hom-Lie-admissible algebras and more general G -Hom-associative algebras with subclasses of Hom-Vinberg and pre-Hom-Lie algebras, generalizing to the twisted situation Lie-admissible algebras, G -associative algebras, Vinberg and pre-Lie algebras respectively, and show that for these classes of algebras the operation of taking commutator leads to Hom-Lie algebras as well. We constructed also all the twistings so that the brackets $[X_1, X_2] = 2X_2$, $[X_1, X_3] = -2X_3$, $[X_2, X_3] = X_1$ determine a three dimensional Hom-Lie algebra. Finally, we provided for a subclass of twistings, the list of all three-dimensional Hom-Lie algebras. This list contains all three-dimensional Lie algebras for some values of structure constants. The families of Hom-Lie algebras in these list can be viewed as deformations of Lie algebras into a class of Hom-Lie algebras. The notion, constructions and properties of the enveloping algebras of Hom-Lie algebras are yet to be properly studied in full generality. An important progress in this direction has been made in the recent work by D. Yau [14].

In the present paper we develop the coalgebra counterpart of the notions and results of [12], extending in particular in the framework of Hom-associative and Hom-Lie algebras and Hom-coalgebras, the notions and results on associative and Lie admissible coalgebras obtained in [2]. In the first section we

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summarize the relevant definitions of Hom-associative algebra, Hom-Lie algebra, Hom-Leibniz algebra, and define the notions of Hom-coalgebras and Hom-coassociative coalgebras. In section 2, we introduce the concept of Hom-Lie admissible Hom-coalgebra, describe some useful relations between coproduct, opposite coproduct, the cocommutator defined as their difference, and their β -twisted coassociators and β -twisted co-Jacobi sums. We also introduce the notion of G -Hom-coalgebra for any subgroup G of permutation group S_3 . We show that G -Hom-coalgebras are Hom-Lie admissible Hom-coalgebras, and also establish duality based correspondence between classes of G -Hom-coalgebras and G -Hom-algebras. The last section is dedicated to relevant definitions and basic properties of the Hom-Hopf algebra which generalize the classical Hopf algebra structure. We also define the module and comodule structure over Hom-associative algebra or Hom-coassociative coalgebra.

1. HOM-ALGEBRA AND HOM-COALGEBRA STRUCTURES

A Hom-algebra structure is a multiplication on a vector space where the structure is twisted by a homomorphism. The structure of Hom-Lie algebra was introduced by Hartwig, Larson and Silvestrov in [4]. In the following we summarize the definitions of Hom-associative, Hom-Leibniz, and Hom-Lie-admissible algebraic structures introduced in [12] and generalizing the well known associative, Leibniz and Lie-admissible algebras. By dualization of Hom-associative algebra we define the Hom-coassociative coalgebra structure.

1.1. Hom-algebra structures. Let \mathbb{K} be an algebraically closed field of characteristic 0 and V be a vector space over \mathbb{K} .

Definition 1.1. A *Hom-associative algebra* over V is a linear map $\mu : V \otimes V \rightarrow V$ and a homomorphism α satisfying

$$(1.1) \quad \mu(\alpha(x) \otimes \mu(y \otimes z)) = \mu(\mu(x \otimes y) \otimes \alpha(z)).$$

The Hom-associativity condition (1.1) may be expressed by the following commutative diagram.

$$\begin{array}{ccc} V \otimes V \otimes V & \xrightarrow{\mu \otimes \alpha} & V \otimes V \\ \downarrow \alpha \otimes \mu & & \downarrow \mu \\ V \otimes V & \xrightarrow{\mu} & V \end{array}$$

The Hom-associative algebra is unital if there exists a homomorphism $\eta : \mathbb{K} \rightarrow V$ such that the following diagrams are commutative

$$\begin{array}{ccccc} \mathbb{K} \otimes V & \xrightarrow{\eta \otimes id} & V \otimes V & \xleftarrow{id \otimes \eta} & V \otimes \mathbb{K} \\ & \searrow \cong & \downarrow \mu & \swarrow \cong & \\ & & V & & \end{array}$$

In the language of Hopf algebra, a Hom-associative algebra \mathcal{A} is a quadruple (V, μ, α, η) where V is the vector space, μ is the Hom-associative multiplication, α is the twisting homomorphism and η is the unit. Let (V, μ, α, η) and $(V', \mu', \alpha', \eta')$ be two Hom-associative algebras. A linear map $f : V \rightarrow V'$ is a morphism of Hom-associative algebras if

$$\mu' \circ (f \otimes f) = f \circ \mu \quad , \quad f \circ \eta = \eta' \quad \text{and} \quad f \circ \alpha = \alpha' \circ f.$$

In particular, (V, μ, α, η) and $(V', \mu', \alpha', \eta')$ are isomorphic if there exists a bijective linear map f such that

$$\mu = f^{-1} \circ \mu' \circ (f \otimes f) \quad , \quad \eta = f^{-1} \circ \eta' \quad \text{and} \quad \alpha = f^{-1} \circ \alpha' \circ f.$$

The tensor product of two Hom-associative algebras $(V_1, \mu_1, \alpha_1, \eta_1)$ and $(V_2, \mu_2, \alpha_2, \eta_2)$ is defined in an obvious way by the Hom-associative algebra $(V_1 \otimes V_2, \mu_1 \otimes \mu_2, \alpha_1 \otimes \alpha_2, \eta_1 \otimes \eta_2)$.

The Hom-Lie algebras were initially introduced by Hartwig, Larson and Silvestrov in [4] motivated initially by examples of deformed Lie algebras coming from twisted discretizations of vector fields.

Definition 1.2. A *Hom-Lie algebra* is a triple $(V, [\cdot, \cdot], \alpha)$ consisting of a linear space V , bilinear map $[\cdot, \cdot] : V \times V \rightarrow V$ and a linear space homomorphism $\alpha : V \rightarrow V$ satisfying

$$\begin{aligned} [x, y] &= -[y, x] \quad (\text{skew-symmetry}) \\ \circlearrowleft_{x,y,z} [\alpha(x), [y, z]] &= 0 \quad (\text{Hom-Jacobi condition}) \end{aligned}$$

for all x, y, z from V , where $\circlearrowleft_{x,y,z}$ denotes summation over the cyclic permutation on x, y, z .

In a similar way we have the following definition of Hom-Leibniz algebra.

Definition 1.3. A *Hom-Leibniz algebra* is a triple $(V, [\cdot, \cdot], \alpha)$ consisting of a linear space V , bilinear map $[\cdot, \cdot] : V \times V \rightarrow V$ and a homomorphism $\alpha : V \rightarrow V$ satisfying

$$(1.2) \quad [[x, y], \alpha(z)] = [[x, z], \alpha(y)] + [\alpha(x), [y, z]].$$

Note that if a Hom-Leibniz algebra is skewsymmetric then it is a Hom-Lie algebra.

1.2. Hom-Coalgebra structures.

Definition 1.4. A *Hom-coassociative coalgebra* is a quadruple $(V, \Delta, \beta, \varepsilon)$ where V is a \mathbb{K} -vector space and

$$\Delta : V \rightarrow V \otimes V, \quad \beta : V \rightarrow V \quad \text{and} \quad \varepsilon : V \rightarrow \mathbb{K}$$

are linear maps satisfying the following conditions:

$$\begin{aligned} (C1) \quad & (\beta \otimes \Delta) \circ \Delta = (\Delta \otimes \beta) \circ \Delta \\ (C2) \quad & (id \otimes \varepsilon) \circ \Delta = id \quad \text{and} \quad (\varepsilon \otimes id) \circ \Delta = id. \end{aligned}$$

The condition (C1) expresses the Hom-coassociativity of the comultiplication Δ . Also, it is equivalent to the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\Delta} & V \otimes V \\ \downarrow \Delta & & \downarrow \beta \otimes \Delta \\ V \otimes V & \xrightarrow{\Delta \otimes \beta} & V \otimes V \otimes V \end{array}$$

The condition (C2) expresses that ε is the counit which is also equivalent to the following commutative diagrams:

$$\begin{array}{ccccc} \mathbb{K} \otimes V & \xleftarrow{\varepsilon \otimes id_V} & V \otimes V & \xrightarrow{id \otimes \varepsilon} & V \otimes \mathbb{K} \\ & \nwarrow \cong & \uparrow \Delta & \nearrow \cong & \\ & & V & & \end{array}$$

Let $(V, \Delta, \beta, \varepsilon)$ and $(V', \Delta', \beta', \varepsilon')$ be two Hom-coassociative coalgebras. A linear map $f : V \rightarrow V'$ is a morphism of Hom-coassociative coalgebras if

$$(f \otimes f) \circ \Delta = \Delta' \circ f, \quad \varepsilon = \varepsilon' \circ f \quad \text{and} \quad f \circ \beta = \beta' \circ f.$$

If $V = V'$ the previous Hom-coassociative coalgebras are isomorphic if there exists a bijective linear map $f : V \rightarrow V$ such that

$$\Delta' = (f \otimes f) \circ \Delta \circ f^{-1}, \quad \varepsilon' = \varepsilon \circ f^{-1} \quad \text{and} \quad \beta = f^{-1} \circ \beta' \circ f.$$

In the sequel, we call *Hom-coalgebra* a triple (V, Δ, β) where V is a \mathbb{K} -vector space, Δ is a comultiplication not necessarily coassociative or Hom-coassociative, that is a linear map $\Delta : V \rightarrow V \otimes V$, and β is a linear map $\beta : V \rightarrow V$.

2. HOM-LIE ADMISSIBLE HOM-COALGEBRAS

Let \mathbb{K} be an algebraically closed field of characteristic 0 and V be a vector space over \mathbb{K} . Let (V, Δ, β) be a Hom-coalgebra where $\Delta : V \rightarrow V \otimes V$ and $\beta : V \rightarrow V$ are linear maps and Δ is not necessarily coassociative or Hom-coassociative.

By a β -coassociator of Δ we call a linear map $c_\beta(\Delta)$ defined by

$$c_\beta(\Delta) := (\Delta \otimes \beta) \circ \Delta - (\beta \otimes \Delta) \circ \Delta.$$

Let \mathcal{S}_3 be the symmetric group of order 3. Given $\sigma \in \mathcal{S}_3$, we define a linear map

$$\Phi_\sigma : V^{\otimes 3} \longrightarrow V^{\otimes 3}$$

by

$$\Phi_\sigma(x_1 \otimes x_2 \otimes x_3) = x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes x_{\sigma^{-1}(3)}.$$

Recall that $\Delta^{op} = \tau \circ \Delta$ where τ is the usual flip that is $\tau(x \otimes y) = y \otimes x$.

Definition 2.1. A triple (V, Δ, β) is a *Hom-Lie admissible Hom-coalgebra* if the linear map

$$\Delta_L : V \longrightarrow V \otimes V$$

defined by $\Delta_L = \Delta - \Delta^{op}$, is a Hom-Lie coalgebra multiplication, that is the following condition is satisfied

$$(2.1) \quad c_\beta(\Delta_L) + \Phi_{(213)} \circ c_\beta(\Delta_L) + \Phi_{(231)} \circ c_\beta(\Delta_L) = 0$$

where (213) and (231) are the two cyclic permutations of order 3 in \mathcal{S}_3 .

Remark 2.2. Since $\Delta_L = \Delta - \Delta^{op}$, the equality $\Delta_L^{op} = -\Delta_L$ holds.

Lemma 2.3. Let (V, Δ, β) be a Hom-coalgebra where $\Delta : V \rightarrow V \otimes V$ and $\beta : V \rightarrow V$ are linear maps and Δ is not necessarily coassociative or Hom-coassociative, then the following relations are true

$$(2.2) \quad c_\beta(\Delta^{op}) = -\Phi_{(13)} \circ c_\beta(\Delta)$$

$$(2.3) \quad (\beta \otimes \Delta^{op}) \circ \Delta = \Phi_{(13)} \circ (\Delta \otimes \beta) \circ \Delta^{op}$$

$$(2.4) \quad (\beta \otimes \Delta) \circ \Delta^{op} = \Phi_{(13)} \circ (\Delta^{op} \otimes \beta) \circ \Delta$$

$$(2.5) \quad (\Delta \otimes \beta) \circ \Delta^{op} = \Phi_{(213)} \circ (\beta \otimes \Delta) \circ \Delta$$

$$(2.6) \quad (\Delta^{op} \otimes \beta) \circ \Delta = \Phi_{(12)} \circ (\Delta \otimes \beta) \circ \Delta.$$

Lemma 2.4. The β -coassociator of Δ_L is expressed using Δ and Δ^{op} as follows:

$$\begin{aligned} (2.7) \quad c_\beta(\Delta_L) &= c_\beta(\Delta) + c_\beta(\Delta^{op}) \\ &\quad - (\Delta \otimes \beta) \circ \Delta^{op} - (\Delta^{op} \otimes \beta) \circ \Delta + \\ &\quad \Phi_{(13)} \circ (\Delta \otimes \beta) \circ \Delta^{op} + \Phi_{(13)} \circ (\Delta^{op} \otimes \beta) \circ \Delta \\ (2.8) \quad &= c_\beta(\Delta) - \Phi_{(13)} \circ c_\beta(\Delta) \\ &\quad - \Phi_{(213)} \circ (\beta \otimes \Delta) \circ \Delta - \Phi_{(12)} \circ (\Delta \otimes \beta) \circ \Delta \\ &\quad + \Phi_{(231)} \circ (\beta \otimes \Delta) \circ \Delta + \Phi_{(231)} \circ (\Delta \otimes \beta) \circ \Delta. \end{aligned}$$

Proposition 2.5. Let (V, Δ, β) be a Hom-coalgebra. Then one has

$$(2.9) \quad c_\beta(\Delta_L) + \Phi_{(213)} \circ c_\beta(\Delta_L) + \Phi_{(231)} \circ c_\beta(\Delta_L) = 2 \sum_{\sigma \in \mathcal{S}_3} (-1)^{\epsilon(\sigma)} \Phi_\sigma \circ c_\beta(\Delta)$$

where $(-1)^{\epsilon(\sigma)}$ is the signature of the permutation σ .

Proof. By (2.8) and multiplication rules in the group S_3 , it follows that

$$\begin{aligned}
 \Phi_{(213)} \circ \mathbf{c}_\beta(\Delta_L) &= \Phi_{(213)} \circ \mathbf{c}_\beta(\Delta) - \Phi_{(213)} \circ \Phi_{(13)} \circ \mathbf{c}_\beta(\Delta) \\
 &\quad - \Phi_{(213)} \circ \Phi_{(213)} \circ (\beta \otimes \Delta) \circ \Delta - \Phi_{(213)} \circ \Phi_{(12)} \circ (\Delta \otimes \beta) \circ \Delta \\
 &\quad + \Phi_{(213)} \circ \Phi_{(23)} \circ (\beta \otimes \Delta) \circ \Delta + \Phi_{(213)} \circ \Phi_{(231)} \circ (\Delta \otimes \beta) \circ \Delta \\
 (2.10) \quad &= \Phi_{(213)} \circ \mathbf{c}_\beta(\Delta) - \Phi_{(12)} \circ \mathbf{c}_\beta(\Delta) \\
 &\quad - \Phi_{(231)} \circ (\beta \otimes \Delta) \circ \Delta - \Phi_{(23)} \circ (\Delta \otimes \beta) \circ \Delta \\
 &\quad + \Phi_{(13)} \circ (\beta \otimes \Delta) \circ \Delta + (\Delta \otimes \beta) \circ \Delta,
 \end{aligned}$$

$$\begin{aligned}
 \Phi_{(231)} \circ \mathbf{c}_\beta(\Delta_L) &= \Phi_{(231)} \circ \mathbf{c}_\beta(\Delta) - \Phi_{(231)} \circ \Phi_{(13)} \circ \mathbf{c}_\beta(\Delta) \\
 &\quad - \Phi_{(231)} \circ \Phi_{(213)} \circ (\beta \otimes \Delta) \circ \Delta - \Phi_{(231)} \circ \Phi_{(12)} \circ (\Delta \otimes \beta) \circ \Delta \\
 &\quad + \Phi_{(231)} \circ \Phi_{(23)} \circ (\beta \otimes \Delta) \circ \Delta + \Phi_{(231)} \circ \Phi_{(231)} \circ (\Delta \otimes \beta) \circ \Delta \\
 (2.11) \quad &= \Phi_{(231)} \circ \mathbf{c}_\beta(\Delta) - \Phi_{(23)} \circ \mathbf{c}_\beta(\Delta) \\
 &\quad - (\beta \otimes \Delta) \circ \Delta - \Phi_{(13)} \circ (\Delta \otimes \beta) \circ \Delta \\
 &\quad + \Phi_{(12)} \circ (\beta \otimes \Delta) \circ \Delta + \Phi_{(213)} \circ (\Delta \otimes \beta) \circ \Delta.
 \end{aligned}$$

After summing up the equalities (2.8), (2.10) and (2.11) the terms on the right hand sides may be pairwise combined into the terms of the form $(-1)^{\epsilon(\sigma)} \Phi_\sigma \circ \mathbf{c}_\beta(\Delta)$ with each one being present in the sum twice for all $\sigma \in S_3$. \square

Definition 2.1 together with (2.9) yields the following corollary.

Corollary 2.6. *A triple (V, Δ, β) is a Hom-Lie admissible Hom-coalgebra if and only if*

$$\sum_{\sigma \in S_3} (-1)^{\epsilon(\sigma)} \Phi_\sigma \circ \mathbf{c}_\beta(\Delta) = 0$$

where $(-1)^{\epsilon(\sigma)}$ is the signature of the permutation σ .

2.1. G -Hom-Coalgebra structures. In this section we introduce, as in the multiplication case, the notion of G -Hom-coalgebra where G is a subgroup of the symmetric group S_3 .

Definition 2.7. Let G be a subgroup of the symmetric group S_3 , A Hom-coalgebra (V, Δ, β) is called G -Hom-coalgebra if

$$(2.12) \quad \sum_{\sigma \in G} (-1)^{\epsilon(\sigma)} \Phi_\sigma \circ \mathbf{c}_\beta(\Delta) = 0$$

where $(-1)^{\epsilon(\sigma)}$ is the signature of the permutation σ .

Proposition 2.8. *Let G be a subgroup of the permutations group S_3 . Then any G -Hom-Coalgebra (V, Δ, β) is a Hom-Lie admissible Hom-coalgebra.*

Proof. The skew-symmetry follows straightaway from the definition. Take the set of conjugacy classes $\{gG\}_{g \in I}$ where $I \subseteq G$, and for any $\sigma_1, \sigma_2 \in I, \sigma_1 \neq \sigma_2 \Rightarrow \sigma_1 G \cap \sigma_2 G = \emptyset$. Then

$$\sum_{\sigma \in S_3} (-1)^{\epsilon(\sigma)} \Phi_\sigma \circ \mathbf{c}_\beta(\Delta) = \sum_{\sigma_1 \in I} \sum_{\sigma_2 \in \sigma_1 G} (-1)^{\epsilon(\sigma)} \Phi_\sigma \circ \mathbf{c}_\beta(\Delta) = 0.$$

\square

The subgroups of S_3 are

$$\begin{aligned}
 G_1 &= \{Id\}, \quad G_2 = \{Id, \tau_{12}\}, \quad G_3 = \{Id, \tau_{23}\}, \\
 G_4 &= \{Id, \tau_{13}\}, \quad G_5 = A_3, \quad G_6 = S_3,
 \end{aligned}$$

where A_3 is the alternating group and where τ_{ij} is the transposition between i and j . We obtain the following type of Hom-Lie-admissible Hom-coalgebras.

- The G_1 -Hom-coalgebras are the Hom-associative coalgebras defined above.

- The G_2 -Hom-coalgebras satisfy the condition

$$\mathbf{c}_\beta(\Delta) + \Phi_{(12)}\mathbf{c}_\beta(\Delta) = 0.$$

- The G_3 -Hom-coalgebras satisfy the condition

$$\mathbf{c}_\beta(\Delta) + \Phi_{(23)}\mathbf{c}_\beta(\Delta) = 0.$$

- The G_4 -Hom-coalgebras satisfy the condition

$$\mathbf{c}_\beta(\Delta) + \Phi_{(13)}\mathbf{c}_\beta(\Delta) = 0.$$

- The G_5 -Hom-coalgebras satisfy the condition

$$\mathbf{c}_\beta(\Delta) + \Phi_{(213)}\mathbf{c}_\beta(\Delta) + \Phi_{(231)}\mathbf{c}_\beta(\Delta) = 0.$$

If the product μ is skewsymmetric then the previous condition is exactly the Hom-Jacobi identity.

- The G_6 -Hom-coalgebras are the Hom-Lie-admissible coalgebras.

The G_2 -Hom-coalgebras may be called Vinberg-Hom-coalgebra and G_3 -Hom-coalgebras may be called preLie-Hom-coalgebras.

Definition 2.9. A *Vinberg-Hom-coalgebra* is a triple (V, Δ, β) consisting of a linear space V , a linear map $\mu : V \rightarrow V \times V$ and a homomorphism β satisfying

$$\mathbf{c}_\beta(\Delta) + \Phi_{(12)}\mathbf{c}_\beta(\Delta) = 0.$$

Definition 2.10. A *preLie-Hom-coalgebra* is a triple (V, Δ, β) consisting of a linear space V , a linear map $\mu : V \rightarrow V \times V$ and a homomorphism β satisfying

$$\mathbf{c}_\beta(\Delta) + \Phi_{(23)}\mathbf{c}_\beta(\Delta) = 0.$$

More generally, by dualization we have a correspondence between G -Hom-associative algebras introduced in [12] and G -Hom-coalgebras for a subgroup G of \mathcal{S}_3 .

Let G be a subgroup of \mathcal{S}_3 and (V, μ, α) be a G -Hom-associative algebra that is $\mu : V \otimes V \rightarrow V$ and $\alpha : V \rightarrow V$ are linear maps and the following condition is satisfied

$$(2.13) \quad \sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} a_{\alpha, \mu} \circ \Phi_\sigma = 0.$$

where $a_{\alpha, \mu}$ is the α -associator that is $a_{\alpha, \mu} = \mu \circ (\mu \otimes \alpha) - \mu \circ (\alpha \otimes \mu)$

Setting

$$(\mu \otimes \alpha)_G = \sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} (\mu \otimes \alpha) \circ \Phi_\sigma \quad \text{and} \quad (\alpha \otimes \mu)_G = \sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} (\alpha \otimes \mu) \circ \Phi_\sigma$$

the condition (2.13) is equivalent to the following commutative diagram

$$\begin{array}{ccc} V \otimes V \otimes V & \xrightarrow{(\mu \otimes \alpha)_G} & V \otimes V \\ \downarrow (\alpha \otimes \mu)_G & & \downarrow \mu \\ V \otimes V & \xrightarrow{\mu} & V \end{array}$$

By the dualization of the square one may obtain the following commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\Delta} & V \otimes V \\ \downarrow \Delta & & \downarrow (\beta \otimes \Delta)_G \\ V \otimes V & \xrightarrow{(\Delta \otimes \beta)_G} & V \otimes V \otimes V \end{array}$$

where

$$(\beta \otimes \Delta)_G = \sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \Phi_\sigma \circ (\beta \otimes \Delta) \quad \text{and} \quad (\Delta \otimes \beta)_G = \sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \Phi_\sigma \circ (\Delta \otimes \beta).$$

The previous commutative diagram expresses that (V, Δ, β) is a G -Hom-coalgebra. More precisely we have the following connection between G -Hom-coalgebras and G -Hom-associative algebras.

Proposition 2.11. Let (V, Δ, β) be a G -Hom-coalgebra where G is a subgroup of \mathcal{S}_3 . Its dual vector space V^* is provided with a G -Hom-associative algebra (V^*, Δ^*, β^*) where Δ^*, β^* are the transpose map.

Proof. Let (V, Δ, β) be a G -Hom-coalgebra. Let V^* be the dual space of V ($V^* = \text{Hom}(V, \mathbb{K})$). Consider the map

$$\begin{aligned} \lambda_n : (V^*)^{\otimes n} &\longrightarrow (V^*)^{\otimes n} \\ f_1 \otimes \cdots \otimes f_n &\longrightarrow \lambda_n(f_1 \otimes \cdots \otimes f_n) \end{aligned}$$

such that for $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$

$$\lambda_n(f_1 \otimes \cdots \otimes f_n)(v_1 \otimes \cdots \otimes v_n) = f_1(v_1) \otimes \cdots \otimes f_n(v_n)$$

and set

$$\mu := \Delta^* \circ \lambda_2 \quad \alpha := \beta^*$$

where the star \star denotes the transpose linear map. Then, the quadruple (V^*, μ, η, α) is a G -Hom-associative algebra. Indeed, $\mu(f_1, f_2) = \mu_{\mathbb{K}} \circ \lambda_2(f_1 \otimes f_2) \circ \Delta$ where $\mu_{\mathbb{K}}$ is the multiplication of \mathbb{K} and $f_1, f_2 \in V^*$. One has

$$\begin{aligned} \mu \circ (\mu \otimes \alpha)(f_1 \otimes f_2 \otimes f_3) &= \mu(\mu(f_1 \otimes f_2) \otimes \alpha(f_3)) \\ &= \mu_{\mathbb{K}} \circ \lambda_2(\mu(f_1 \otimes f_2) \otimes \alpha(f_3)) \circ \Delta \\ &= \mu_{\mathbb{K}} \circ \lambda_2(\lambda_2((f_1 \otimes f_2) \circ \Delta) \otimes \alpha(f_3)) \circ \Delta \\ &= \mu_{\mathbb{K}} \circ (\mu_{\mathbb{K}} \otimes id) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (\Delta \otimes \beta) \circ \Delta. \end{aligned}$$

Similarly

$$\mu \circ (\alpha \otimes \mu)(f_1 \otimes f_2 \otimes f_3) = \mu_{\mathbb{K}} \circ (id \otimes \mu_{\mathbb{K}}) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (\beta \otimes \Delta) \circ \Delta.$$

Using the associativity and the commutativity of $\mu_{\mathbb{K}}$, the α -associator may be written as

$$a_{\alpha, \mu} = \mu_{\mathbb{K}} \circ (id \otimes \mu_{\mathbb{K}}) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ ((\Delta \otimes \beta) \circ \Delta - (\beta \otimes \Delta) \circ \Delta).$$

Then we have the following connection between the α -associator and β -coassociator

$$a_{\alpha, \mu} = \mu_{\mathbb{K}} \circ (id \otimes \mu_{\mathbb{K}}) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ c_{\beta}(\Delta).$$

Therefore if (V, Δ, β) is a G -Hom-coalgebra, then the (V^*, Δ^*, β^*) is a G -Hom-associative algebra. \square

Proposition 2.12. *Let (V, μ, α) be a finite dimensional G -Hom-associative algebra where G is a subgroup of S_3 . Its dual vector space V^* is provided with a G -Hom-coalgebra (V^*, μ^*, α^*) , where μ^*, α^* are the transpose map.*

Proof. Let $\mathcal{A} = (V, \mu, \alpha)$ be a n -dimensional Hom-associative algebra (n finite). Let $\{e_1, \dots, e_n\}$ be a basis of V and $\{e_1^*, \dots, e_n^*\}$ be the dual basis. Then $\{e_i^* \otimes e_j^*\}_{i,j}$ is a basis of $\mathcal{A}^* \otimes \mathcal{A}^*$. The comultiplication $\Delta = \mu^*$ on \mathcal{A}^* is defined for $f \in \mathcal{A}^*$ by

$$\Delta(f) = \sum_{i,j=1}^n f(\mu(e_i \otimes e_j)) e_i^* \otimes e_j^*$$

Setting $\mu(e_i \otimes e_j) = \sum_{k=1}^n C_{ij}^k e_k$ and $\alpha(e_i) = \sum_{k=1}^n \alpha_i^k e_k$, then $\Delta(e_k^*) = \sum_{i,j=1}^n C_{ij}^k e_i^* \otimes e_j^*$ and $\beta(e_i) = \alpha^*(e_i) = \sum_{k=1}^n \alpha_k^i e_k$.

The condition (2.12) of G -Hom-coassociativity of Δ , applied to any element e_k^* of the basis, is equivalent to

$$\sum_{p,q,s=1}^n \sum_{\sigma \in G} (-1)^{\epsilon(\sigma)} \left(\sum_{i,j=1}^n \alpha_s^j C_{ij}^k C_{pq}^i - \alpha_p^i C_{ij}^k C_{qs}^j \right) e_{\sigma^{-1}(p)}^* \otimes e_{\sigma^{-1}(q)}^* \otimes e_{\sigma^{-1}(s)}^* = 0$$

Therefore Δ is G -Hom-coassociative if for any $p, q, s, k \in \{1, \dots, n\}$ one has

$$\sum_{\sigma \in G} (-1)^{\epsilon(\sigma)} \left(\sum_{i,j=1}^n \alpha_s^j C_{ij}^k C_{pq}^i - \alpha_p^i C_{ij}^k C_{qs}^j \right) = 0$$

The previous system is exactly the condition (2.13) of G -Hom-associativity of μ , written on $e_{p'} \otimes e_{q'} \otimes e_{s'}$ and setting $p = \sigma(p')$, $q = \sigma(q')$, $s = \sigma(s')$. \square

Corollary 2.13. *The dual vector space of a Hom-coassociative coalgebra $(V, \Delta, \beta, \varepsilon)$ is a Hom-associative algebra $(V^*, \Delta^*, \beta^*, \varepsilon^*)$, where V^* is the dual vector space and the star for the linear maps denotes the transpose map. The dual vector space of finite-dimensional Hom-associative algebra is a Hom-coassociative coalgebra.*

Proof. It is a particular case of the previous Propositions ($G = G_1$). \square

3. HOM-HOPF ALGEBRAS

In this section, we introduce a generalization of Hopf algebras and show some relevant properties of the new structure. We also define the module and comodule structure over Hom-associative algebra or Hom-coassociative coalgebra. Let \mathbb{K} be an algebraically closed field of characteristic 0 and V be a vector space over \mathbb{K} .

Definition 3.1. A Hom-bialgebra is a quintuple $(V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ where

- (B1) (V, μ, α, η) is a Hom-associative algebra
- (B2) $(V, \Delta, \beta, \varepsilon)$ is a Hom-coassociative coalgebra
- (B3) The linear maps Δ and ε are morphisms of algebras (V, μ, α, η) .

Remark 3.2. The condition (B3) could be expressed by the following system:

$$\begin{cases} \Delta(e_1) = e_1 \otimes e_1 & \text{where } e_1 = \eta(1) \\ \Delta(\mu(x \otimes y)) = \Delta(x) \bullet \Delta(y) = \sum_{(x)(y)} \mu(x^{(1)} \otimes y^{(1)}) \otimes \mu(x^{(2)} \otimes y^{(2)}) \\ \varepsilon(e_1) = 1 \\ \varepsilon(\mu(x \otimes y)) = \varepsilon(x) \varepsilon(y) \end{cases}$$

where the bullet \bullet denotes the multiplication on tensor product and by using the Sweedler's notation $\Delta(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)}$. If there is no ambiguity we denote the multiplication by a dot.

Remark 3.3. One can consider a more restrictive definition where linear maps Δ and ε are morphisms of Hom-associative algebras that is the condition (B3) becomes equivalent to

$$\begin{cases} \Delta(e_1) = e_1 \otimes e_1 & \text{where } e_1 = \eta(1) \\ \Delta(\mu(x \otimes y)) = \Delta(x) \bullet \Delta(y) = \sum_{(x)(y)} \mu(x^{(1)} \otimes y^{(1)}) \otimes \mu(x^{(2)} \otimes y^{(2)}) \\ \varepsilon(e_1) = 1 \\ \varepsilon(\mu(x \otimes y)) = \varepsilon(x) \varepsilon(y) \\ \Delta(\alpha(x)) = \sum_{(x)} \alpha(x^{(1)}) \otimes \alpha(x^{(2)}) \\ \varepsilon \circ \alpha(x) = \varepsilon(x) \end{cases}$$

Given a Hom-bialgebra $(V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$, we show that the vector space $\text{Hom}(V, V)$ with the multiplication given by the convolution product carries a structure of Hom-algebra.

Proposition 3.4. *Let $(V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ be a Hom-bialgebra. Then the algebra $\text{Hom}(V, V)$ with the multiplication given by the convolution product defined by*

$$f \star g = \mu \circ (f \otimes g) \circ \Delta$$

and the unit being $\eta \circ \varepsilon$ is a Hom-associative algebra with the homomorphism map defined by $\gamma(f) = \alpha \circ f \circ \beta$.

Proof. Let $f, g, h \in \text{Hom}(V, V)$.

$$\begin{aligned} \gamma(f) \star (g \star h) &= \mu \circ (\gamma(f) \otimes (g \star h)) \Delta \\ &= \mu \circ (\gamma(f) \otimes (\mu \circ (g \otimes h) \circ \Delta)) \Delta \\ &= \mu \circ (\alpha \otimes \mu) \circ (f \otimes g \otimes h) \circ (\beta \otimes \Delta) \Delta. \end{aligned}$$

Similarly

$$(f \star g) \star \gamma(h) = \mu \circ (\mu \otimes \alpha) \circ (f \otimes g \otimes h) \circ (\Delta \otimes \beta) \Delta.$$

Then, the Hom-associativity of μ and a Hom-coassociativity of Δ lead to the Hom-associativity of the convolution product. The unitality is as usual. \square

Definition 3.5. An endomorphism S of V is said to be an *antipode* if it is the inverse of the identity over V for the Hom-algebra $\text{Hom}(V, V)$ with the multiplication given by the convolution product defined by

$$f \star g = \mu \circ (f \otimes g) \Delta$$

and the unit being $\eta \circ \epsilon$.

The condition being antipode may be expressed by the condition:

$$\mu \circ S \otimes \text{Id} \circ \Delta = \mu \circ \text{Id} \otimes S \circ \Delta = \eta \circ \epsilon.$$

Definition 3.6. A *Hom-Hopf algebra* is a Hom-bialgebra with an antipode.

Then, a Hom-Hopf algebra over a \mathbb{K} -vector space V is given by

$$\mathcal{H} = (V, \mu, \alpha, \eta, \Delta, \beta, \epsilon, S)$$

where the following homomorphisms

$$\begin{aligned} \mu : V \otimes V &\rightarrow, & \eta : \mathbb{K} &\rightarrow V, & \alpha : V &\rightarrow V \\ \Delta : V &\rightarrow V \otimes V, & \epsilon : V &\rightarrow \mathbb{K}, & \beta : V &\rightarrow V \\ S : V &\rightarrow \mathbb{K} \end{aligned}$$

satisfy the following conditions

- (1) (V, μ, α, η) is a unital Hom-associative algebra.
- (2) $(V, \Delta, \beta, \epsilon)$ is a counital Hom-coalgebra.
- (3) Δ and ϵ are morphisms of algebras, which translate to

$$\begin{cases} \Delta(e_1) = e_1 \otimes e_1 & \text{where } e_1 = \eta(1) \\ \Delta(x \cdot y) = \Delta(x) \bullet \Delta(y) = \sum_{(x)(y)} x^{(1)} \cdot y^{(1)} \otimes x^{(2)} \cdot y^{(2)} \\ \epsilon(e_1) = 1 \\ \epsilon(x \cdot y) = \epsilon(x) \epsilon(y) \end{cases}$$

- (4) S is the antipode, so

$$\mu \circ S \otimes \text{Id} \circ \Delta = \mu \circ \text{Id} \otimes S \circ \Delta = \eta \circ \epsilon.$$

Remark 3.7. Let V be a finite-dimensional \mathbb{K} -vector space. If $H = (V, \mu, \alpha, \eta, \Delta, \beta, \epsilon, S)$ is a Hom-Hopf algebra, then

$$H^* = (V^*, \Delta^*, \beta^*, \epsilon^*, \mu^*, \alpha^*, \eta^*, S^*)$$

is also a Hom-Hopf algebra.

3.1. Primitive elements and Generalized Primitive elements. In the following, we discuss the properties of primitive elements in a Hom-bialgebra.

Let $\mathcal{H} = (V, \mu, \alpha, \eta, \Delta, \beta, \epsilon)$ be a Hom-bialgebra and $e_1 = \eta(1)$ be the unit.

Definition 3.8. An element $x \in \mathcal{H}$ is called primitive if $\Delta(x) = e_1 \otimes x + x \otimes e_1$.

Let $x \in \mathcal{H}$ be a primitive element, the coassociativity of Δ implies

$$(\beta \otimes \Delta) \circ \Delta(x) = \tau_{13} \circ (\Delta \otimes \beta) \circ \Delta(x)$$

where τ_{13} is a permutation in the symmetric group \mathcal{S}_3 .

Lemma 3.9. Let x be a primitive element in \mathcal{H} , then $\epsilon(x) = 0$.

Proof. By counity property, we have $x = (\text{id} \otimes \epsilon) \circ \Delta(x)$. If $\Delta(x) = e_1 \otimes x + x \otimes e_1$, then $x = \epsilon(x)e_1 + \epsilon(e_1)x$, and since $\epsilon(e_1) = 1$ it implies $\epsilon(x) = 0$. \square

Proposition 3.10. Let $\mathcal{H} = (V, \mu, \alpha, \eta, \Delta, \beta, \epsilon)$ be a Hom-bialgebra and $e_1 = \eta(1)$ be the unit. If x and y are two primitive elements in \mathcal{H} . Then we have $\epsilon(x) = 0$ and the commutator $[x, y] = \mu(x \otimes y) - \mu(y \otimes x)$ is also a primitive element.

The set of all primitive elements of \mathcal{H} , denoted by $\text{Prim}(\mathcal{H})$, has a structure of Hom-Lie algebra.

Proof. By a direct calculation one has

$$\begin{aligned}
\Delta([x, y]) &= \Delta(\mu(x \otimes y) - \mu(y \otimes x)) \\
&= \Delta(x) \bullet \Delta(y) - \Delta(y) \bullet \Delta(x) \\
&= (e_1 \otimes x + x \otimes e_1) \bullet (e_1 \otimes y + y \otimes e_1) - (e_1 \otimes y + y \otimes e_1) \bullet (e_1 \otimes x + x \otimes e_1) \\
&= e_1 \otimes \mu(x \otimes y) + y \otimes x + x \otimes y + \mu(x \otimes y) \otimes e_1 \\
&\quad - e_1 \otimes \mu(y \otimes x) - x \otimes y - y \otimes x - \mu(y \otimes x) \otimes e_1 \\
&= e_1 \otimes (\mu(x \otimes y) - \mu(y \otimes x)) + (\mu(x \otimes y) - \mu(y \otimes x)) \otimes e_1 \\
&= e_1 \otimes [x, y] + [x, y] \otimes e_1
\end{aligned}$$

which means that $\text{Prim}(\mathcal{H})$ is closed under the bracket multiplication $[\cdot, \cdot]$.

We have seen in [12] that there is a natural map from the Hom-associative algebras to Hom-Lie algebras. The bracket $[x, y] = \mu(x \otimes y) - \mu(y \otimes x)$ is obviously skewsymmetric and one checks that the Hom-Jacobi condition is satisfied:

$$\begin{aligned}
&[\alpha(x), [y, z]] - [[x, y], \alpha(z)] - [\alpha(y), [x, z]] = \\
&\mu(\alpha(x) \otimes \mu(y \otimes z)) - \mu(\alpha(x) \otimes \mu(z \otimes y)) - \mu(\mu(y \otimes z) \otimes \alpha(x)) + \mu(\mu(z \otimes y) \otimes \alpha(x)) \\
&- \mu(\mu(x \otimes y) \otimes \alpha(z)) + \mu(\mu(y \otimes x) \otimes \alpha(z)) + \mu(\alpha(z) \otimes \mu(x \otimes y)) - \mu(\alpha(z) \otimes \mu(y \otimes x)) \\
&- \mu(\alpha(y) \otimes \mu(x \otimes z)) + \mu(\alpha(y) \otimes \mu(z \otimes x)) + \mu(\mu(x \otimes z) \otimes \alpha(y)) - \mu(\mu(z \otimes x) \otimes \alpha(y)) = 0
\end{aligned}$$

□

We introduce now a notion of generalized primitive element.

Definition 3.11. An element $x \in \mathcal{H}$ is called generalized primitive element if it satisfies the conditions

$$(3.1) \quad (\beta \otimes \Delta) \circ \Delta(x) = \tau_{13} \circ (\Delta \otimes \beta) \circ \Delta(x)$$

$$(3.2) \quad \Delta^{op}(x) = \Delta(x)$$

where τ_{13} is a permutation in the symmetric group \mathcal{S}_3 .

Remark 3.12. (1) In particular, a primitive element in \mathcal{H} is a generalized primitive element.

(2) The condition (3.1) may be written

$$(\Delta \otimes \beta) \circ \Delta(x) = \tau_{13} \circ (\beta \otimes \Delta) \circ \Delta(x).$$

Proposition 3.13. Let $\mathcal{H} = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ be a Hom-bialgebra and $e_1 = \eta(1)$ be the unit. If x and y are two generalized primitive elements in \mathcal{H} . Then, we have $\varepsilon(x) = 0$ and the commutator $[x, y] = \mu(x \otimes y) - \mu(y \otimes x)$ is also a generalized primitive element.

The set of all generalized primitive elements of \mathcal{H} , denoted by $G\text{Prim}(\mathcal{H})$, has a structure of Hom-Lie algebra.

Proof. Let x and y be two generalized primitive elements in \mathcal{H} . In the following the multiplication μ is denoted by a dot. The following equalities hold:

$$\begin{aligned}
(\Delta \otimes \beta) \circ \Delta(x \cdot y - y \cdot x) &= (\Delta \otimes \beta) \circ \Delta(x \cdot y) - (\Delta \otimes \beta) \circ \Delta(y \cdot x) \\
&= (\Delta \otimes \beta)(\Delta(x) \bullet \Delta(y)) - (\Delta \otimes \beta)(\Delta(y) \bullet \Delta(x)) \\
&= \Delta(x^{(1)} \cdot y^{(1)}) \otimes \beta(x^{(2)} \cdot y^{(2)}) - \Delta(y^{(1)} \cdot x^{(1)}) \otimes \beta(y^{(2)} \cdot x^{(2)}) \\
&= (x^{(1)(1)} \cdot y^{(1)(1)}) \otimes (x^{(1)(2)} \cdot y^{(1)(2)}) \otimes \beta(x^{(2)} \cdot y^{(2)}) \\
&\quad - (y^{(1)(1)} \cdot x^{(1)(1)}) \otimes (y^{(1)(2)} \cdot x^{(1)(2)}) \otimes \beta(y^{(2)} \cdot x^{(2)}).
\end{aligned}$$

Then, using the fact that $\Delta^{op} = \Delta$ for generalized primitive elements one has:

$$\begin{aligned}
\tau_{13} \circ (\Delta \otimes \beta) \circ \Delta(x \cdot y - y \cdot x) &= \beta(x^{(2)} \cdot y^{(2)}) \otimes (x^{(1)(2)} \cdot y^{(1)(2)}) \otimes (x^{(1)(1)} \cdot y^{(1)(1)}) \\
&\quad - \beta(y^{(2)} \cdot x^{(2)}) \otimes (y^{(1)(2)} \cdot x^{(1)(2)}) \otimes (y^{(1)(1)} \cdot x^{(1)(1)}) \\
&= (\beta \otimes \Delta) \circ \Delta(x \cdot y - y \cdot x).
\end{aligned}$$

The structure of Hom-Lie algebra follows from the same argument as in the primitive elements case. \square

3.2. Antipode's properties. Let $\mathcal{H} = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon, S)$ be a Hom-Hopf algebra. For any element $x \in V$, using the counity and Sweedler notation, one may write

$$(3.3) \quad x = \sum_{(x)} x^{(1)} \otimes \varepsilon(x^{(2)}) = \sum_{(x)} \varepsilon(x^{(1)}) \otimes x^{(2)}.$$

Then, for any $f \in \text{End}_{\mathbb{K}}(V)$, we have

$$(3.4) \quad f(x) = \sum_{(x)} f(x^{(1)}) \varepsilon(x^{(2)}) = \sum_{(x)} \varepsilon(x^{(1)}) \otimes f(x^{(2)}).$$

Let $f \star g = \mu \circ (f \otimes g) \Delta$ be the convolution product of $f, g \in \text{End}_{\mathbb{K}}(V)$. One may write

$$(3.5) \quad (f \star g)(x) = \sum_{(x)} \mu(f(x^{(1)}) \otimes g(x^{(2)})).$$

Since the antipode S is the inverse of the identity for the convolution product then S satisfies

$$(3.6) \quad \varepsilon(x) \eta(1) = \sum_{(x)} \mu(S(x^{(1)}) \otimes x^{(2)}) = \sum_{(x)} \mu(x^{(1)} \otimes S(x^{(2)})).$$

Proposition 3.14. *The antipode S is unique and we have*

- $S(\eta(1)) = \eta(1)$.
- $\varepsilon \circ S = \varepsilon$.

Proof. 1) We have $S \star \text{id} = \text{id} \star S = \eta \circ \varepsilon$. Thus, $(S \star \text{id}) \star S = S \star (\text{id} \star S) = S$. If S' is another antipode of \mathcal{H} then

$$S' = S' \star \text{id} \star S' = S' \star \text{id} \star S = S \star \text{id} \star S = S.$$

Therefore the antipode when it exists is unique.

2) Setting $e_1 = \eta(1)$ and since $\Delta(e_1) = e_1 \otimes e_1$ one has

$$(S \star \text{id})(e_1) = \mu(S(e_1) \otimes e_1) = S(e_1) = \eta(\varepsilon(e_1)) = e_1.$$

3) Applying (3.4) to S , we obtain $S(x) = \sum_{(x)} S(x^{(1)}) \varepsilon(x^{(2)})$.

Applying ε to (3.6), we obtain

$$\varepsilon(x) = \varepsilon\left(\sum_{(x)} \mu(S(x^{(1)}) \otimes x^{(2)})\right).$$

Since ε is a Hom-algebra morphism, one has

$$\varepsilon(x) = \sum_{(x)} \varepsilon(S(x^{(1)})) \varepsilon(x^{(2)}) = \varepsilon\left(\sum_{(x)} S(x^{(1)}) \varepsilon(x^{(2)})\right) = \varepsilon(S(x)).$$

Thus $\varepsilon \circ S = \varepsilon$. \square

3.3. Modules and Comodules. We introduce in the following the structure of module and comodule over Hom-associative algebras.

Let $\mathcal{A} = (V, \mu, \alpha)$ be a Hom-associative \mathbb{K} -algebra, an \mathcal{A} -module (left) is a triple (M, f, γ) where M is \mathbb{K} -vector space and f, γ are \mathbb{K} -linear maps, $f : M \rightarrow M$ and $\gamma : V \otimes M \rightarrow M$, such that the following diagram commutes:

$$\begin{array}{ccc} V \otimes V \otimes M & \xrightarrow{\mu \otimes f} & V \otimes M \\ \downarrow \alpha \otimes \gamma & & \downarrow \gamma \\ V \otimes M & \xrightarrow{\gamma} & M \end{array}$$

The dualization leads to comodule definition over a Hom-coassociative coalgebra.

Let $C = (V, \Delta, \beta)$ be a Hom-coassociative coalgebra. A C -comodule (right) is a triple (M, g, ρ) where M is a \mathbb{K} -vector space and g, ρ are \mathbb{K} -linear maps, $g : M \rightarrow M$ and $\rho : M \rightarrow M \otimes V$, such that the following diagram commutes:

$$\begin{array}{ccc}
M & \xrightarrow{\rho} & M \otimes V \\
\downarrow \rho & & \downarrow g \otimes \Delta \\
M \otimes V & \xrightarrow{\rho \otimes \beta} & M \otimes V \otimes V
\end{array}$$

Remark 3.15. A Hom-associative \mathbb{K} -algebra $\mathcal{A} = (V, \mu, \alpha)$ is a left \mathcal{A} -module with $M = V$, $f = \alpha$ and $\gamma = \mu$. Also, a Hom-coassociative coalgebra $C = (V, \Delta, \beta)$ is a right C -comodule with $M = V$, $g = \beta$ and $\rho = \Delta$. The properties of modules and comodules over Hom-associative algebras or Hom-coassociative algebras will be discussed in a forthcoming paper.

3.4. Examples. The classification of 2-dimensional Hom-associative algebras, up to isomorphism, yields to the following two classes. Let $B = \{e_1, e_2\}$ be a basis where $\eta(1) = e_1$ is the unit.

- (1) The multiplication μ_1 is defined by $\mu_1(e_1 \otimes e_i) = \mu_1(e_i \otimes e_1) = e_i$ for $i = 1, 2$ and $\mu_1(e_2 \otimes e_2) = e_2$ and the homomorphism α_1 is defined, with respect to the basis B by $\begin{pmatrix} a_1 & 0 \\ a_2 - a_1 & a_2 \end{pmatrix}$.
- (2) The multiplication μ_2 is defined by $\mu_2(e_1 \otimes e_i) = \mu_2(e_i \otimes e_1) = e_i$ for $i = 1, 2$ and $\mu_2(e_2 \otimes e_2) = 0$ and the homomorphism α_2 is defined, with respect to the basis B by $\begin{pmatrix} a_1 & 0 \\ a_2 & a_1 \end{pmatrix}$.

The Hom-bialgebras corresponding to the Hom-associative algebra defined by μ_1 and α_1 are given in the following table

	Comultiplication	Co-unit	homomorphism
1	$\Delta(e_1) = e_1 \otimes e_1$ $\Delta(e_2) = e_2 \otimes e_2$	$\varepsilon(e_1) = 1$ $\varepsilon(e_2) = 1$	$\begin{pmatrix} b_1 & 0 \\ b_3 & b_2 \end{pmatrix}$
2	$\Delta(e_1) = e_1 \otimes e_1$ $\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - 2e_2 \otimes e_2$	$\varepsilon(e_1) = 1$ $\varepsilon(e_2) = 0$	$\begin{pmatrix} b_1 & \frac{b_1 - b_3}{2} \\ b_2 & b_3 \end{pmatrix}$
3	$\Delta(e_1) = e_1 \otimes e_1$ $\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - e_2 \otimes e_2$	$\varepsilon(e_1) = 1$ $\varepsilon(e_2) = 0$	$\begin{pmatrix} b_1 & b_1 - b_3 \\ b_2 & b_3 \end{pmatrix}$

Only Hom-bialgebra (2) carries a structure of Hom-Hopf algebra with an antipode defined, with respect to a basis B , by the identity matrix.

Remark 3.16. There is no Hom-bialgebra associated to the Hom-associative algebra defined by the multiplication μ_2 and any homomorphism α_2 .

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